Topics in Rational and Integral Points

Week 1 (September 2 – 6, 2019)

Venue: Lecture hall -101 in Old University, Rheinsprung 9, Basel

	Monday	Tuesday	Wednesday	Thursday	Friday
08.45 - 09.15	*Registration*				
09.15 - 10.00 10.15 - 11.00	**Bilu**	**Schindler**	**Krieger**	**Binyamini**	**Schindler** (08.15-09.00, 09.15-10.00)
	Coffee break				
11.30 - 12.15	**Binyamini**	**Bilu**	**Schindler** (11.30 - 12.15, 12.30 - 13.15)	**Krieger**	**Zannier** (10.30-11.30, 11.45-12.45)
	Lunch break		*Free afternoon*	*Lunch break*	
13.45 - 14.30	**Binyamini**	**Bilu**		**Krieger**	
	Coffee break			*Coffee break*	
15.00 - 15.45 16.00 - 16.45	**Krieger**	**Binyamini**		**Bilu**	
18.00 -	*Welcome Apéro*				

Mini-course titles and abstracts

Yuri Bilu

Effective André-Oort

Abstract: On every Shimura variety certain algebraic subvarieties are called "special"; in particular, "special points" are special subvarieties of dimension 0. Intuitively, if the Shimura variety is viewed as a moduli space of some objects (like Abelian varieties or so), then the special subvarities are moduli spaces of the same objects with some additional structure.

The celebrated André-Oort Conjecture asserts, roughly speaking, that the Zariski closure of a set of "special points" is a "special subvariety"; equivalently, every algebraic subvariety of a Shimura variety may have at most finitely many maximal special subvarieties. This conjecture is proved by Klingler, Ullmo and Yafaev subject to GRH, and in many special cases unconditionally.

In particular, Pila (2011) proved unconditional André-Oort for Shimura varieties of modular type, that is, products of modular curves. Here every point is $(x_1, ..., x_n)$, where each x_i is an elliptic curve with some additional structure. Special subvarieties are (roughly) defined by conditions of the type "there is a cyclic isogeny of given degree between x_i and x_j" or "x_i is a given curve with Complex Multiplication".

While Pila's argument (based on the ground-breaking idea of a previous work of Pila and Zannier) is very clever and beautiful, it is non-effective, though the use of Siegel-Brauer lower estimate for the class number.

In the recent years, in the work of Masser, Zannier, Kühne, Binyamini and others effective proofs were given for various special cases of Pila's result. Most of them are based on the beautiful "Tatuzawa trick", discovered by Kühne, which allows one to achieve effectiveness, in many cases, by replacing the Siegel-Brauer by the Siegel-Tatuzawa Theorem.

The purpose of my course is to give an introduction into this topic. I will restrict to the "Shimura variety" $\mathbb{C}^n$ (viewed as the product of n j-lines). The special subvarieties are (roughly) defined by equations of the type $F_N(x_i, x_j) = 0$ or $x_i =$(singular modulus), where F_N is the modular polynomial of level N, and singular moduli are j-invariants of elliptic curves with CM. In particular, special points are those whose all coordinates are singular moduli.

No preliminary knowledge beyond university course of Algebra, Analysis and Number Theory is required. I will even define what Complex Multiplication is. I will also try to highlight ideas as clearly as possible, in many cases sacrificing generality to lucidity. Here is an approximate plan.

- -1. Motivation: The Manin-Mumford conjecture
- 0. Introductory material: Complex Multiplication, Class Field Theory etc.
- 1. Statement of the André-Oort conjecture for $\mathbb{C}^n$
- 2. Non-effective proof in dimension 1 (André)
- 3. Effectivisation of André's argument (Kühne et al.)
- 4. One-dimensional effective/explicit results on individual curves and families on curves.
- 5. Multidimensional results (if time permits)

Gal Binyamini

Tame geometry meets diophantine geometry

Abstract: Tame geometry studies classes of sets that enjoy good finiteness properties. A powerful formalism of tame geometry is provided by the model theoretic framework of o-minimal structures. This includes the classical classes of semialgebraic and subanalytic sets, and a plethora of larger classes. In 2006 Pila and Wilkie discovered a powerful link between o-minimality and diophantine geometry, known as the counting theorem. It implies a strong asymptotic upper bound for the number of rational points of a tame set as a function of height under a certain transcendence assumption. Over the past decade this theorem has played the key role in the resolution of several outstanding problems in diophantine geometry. In this minicourse I will try to present a self-contained introduction to tame geometry, the counting theorem, and its typical diophantine applications. If time permits I will also discuss some further progress and conjectures around the counting theorem.

Holly Krieger

Equidistribution in arithmetic and dynamics

Abstract: This course will illustrate the theory of equidistribution of small points on curves from the perspective of potential theory. I will focus on the quantitative equidistribution theory of Favre and Rivera-Letelier, and demonstrate applications of these methods to the arithmetic of elliptic curves and the arithmetic of dynamical systems.

Damaris Schindler

Interactions of analytic number theory and geometry

Abstract: The goal of this mini-course is to introduce different techniques from analytic number theory which are useful in the study of rational points on varieties.

We will start with a conjecture of Harpaz and Wittenberg on split values of polynomials, which implies that the Brauer-Manin obstruction is the only obstruction to the Hasse principle and weak approximation for a large class of varieties over number fields.

We will take this as a starting point to discuss different analytic tools with which one can attack cases of this conjecture.

Umberto Zannier

The Schmidt Subspace Theorem and some applications

Abstract: In the first part of this mini-mini-course, after recalling the context and main statements of the Subspace Theorem, we shall describe in a bit of detail some applications to diophantine problems, in particular the S-unit equation (in several variables) and a proof of Siegel's theorem on integral points. We shall also briefly mention some applications of the methods in higher dimensions. In the second part we shall start by presenting other applications, to linear recurrences and gcd problems. Then we shall conclude by discussing briefly some applications to proofs of transcendence.